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absence of arb $\iff \exists\ \mathbb{Q} \sim \mathbb{P}$ s.t.

relative prices w.r.t. numeraire asset

are martingales:

$\forall$ traded assets A, and a numeraire B, we have:

$$\frac{A_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_1}{B_1}\right]$$

$\to \mathbb{P}(B_t > 0) = 1$

$\in (0,1)$

$$A_0 \begin{cases} \xrightarrow{p,\ q} A_u \\ \to A_d \end{cases} \qquad B_0 \begin{cases} \xrightarrow{p,\ q} B_u \\ \to B_d \end{cases} \qquad \left(1 \begin{cases} 1+r \\ 1+r \end{cases} \right)$$

e.g. bank account

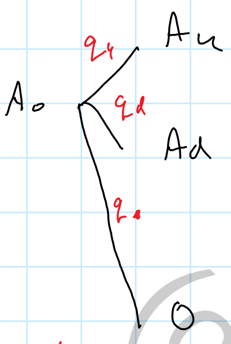
$\mathbb{Q}$ is called the risk-neutral measure.

$$\frac{A_0}{B_0} = \frac{A_u}{B_u} q + \frac{A_d}{B_d}(1-q)$$

$$\Rightarrow \quad q = \frac{\frac{A_0}{B_0} - \frac{A_d}{B_d}}{\frac{A_u}{B_u} - \frac{A_d}{B_d}} \in (0,1)$$

$$\iff \quad \frac{A_d}{B_d} < \frac{A_0}{B_0} < \frac{A_u}{B_u} \quad \text{OR}$$

$$\frac{A_u}{B_u} < \frac{A_0}{B_0} < \frac{A_d}{B_d}$$



α

β

$A_0 = q_u A_u + q_d A_d + q_0\, 0$

$1 = q_u + q_d + q_0$

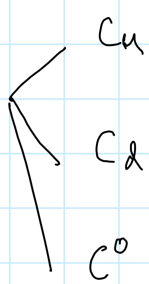
$q_u, q_d, q_0 \geq 0$

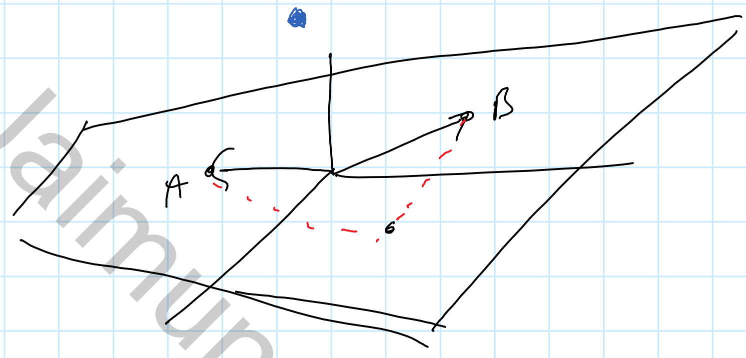
$\underline{\alpha \text{ is not unique}}$

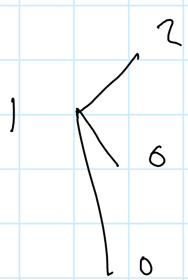
$? = C_0$

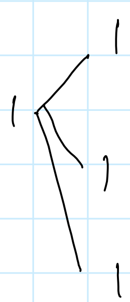
$\parallel$

$\alpha A_0 + \beta$









① $1 = 2q_u + 0\, q_d + 0\, q_0$

② $1 = q_u + q_d + q_0$

③ $q_u, q_d, q_0 \geq 0$

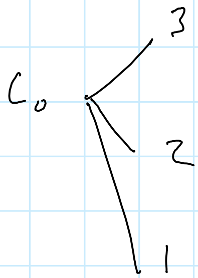
① $\Rightarrow q_u = 1/2$

② $\Rightarrow 1 = \frac{1}{2} + q_d + q_0 \Rightarrow q_d + q_0 = \frac{1}{2}$

$q_d = \frac{1}{2} - x, \quad q_0 = x$

③ $\Rightarrow q_0 = x > 0$

$q_d = \frac{1}{2} - x > 0$



③ $\Rightarrow$ $q_0 = x > 0$

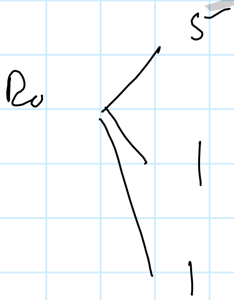
$q_d = \frac{1}{2} - x > 0$

$\Rightarrow \quad x \in (0, \frac{1}{2})$

$$C_0 = 3\, q_u + 2\, q_d + 1\, q_0$$

$$= \frac{3}{2} + 2\left(\frac{1}{2} - x\right) + x$$

$$= \frac{5}{2} - x \quad \in \left(2, 2\tfrac{1}{2}\right)$$

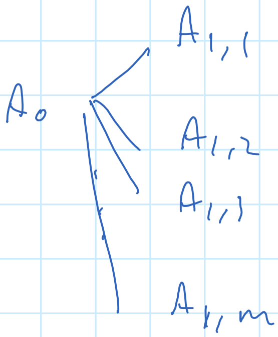


$$D_0 = 5\,\tfrac{1}{2} + 1\left(\tfrac{1}{2} - x\right) + x$$

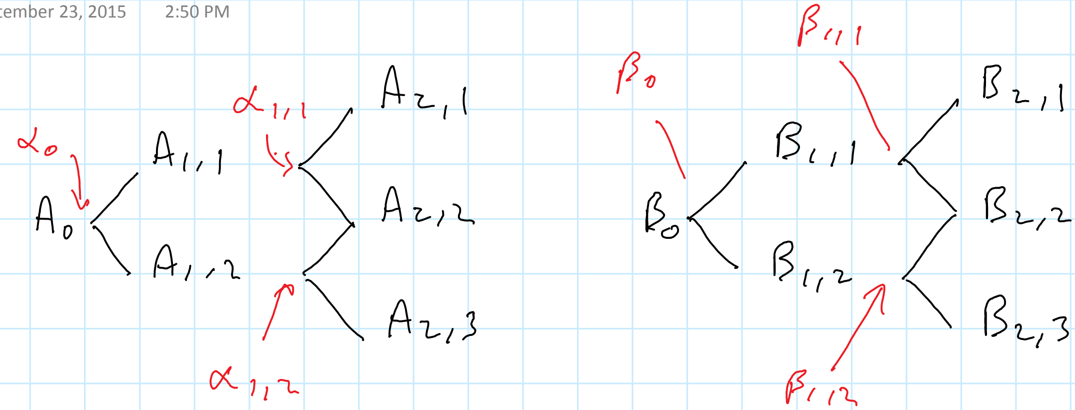
$$= 3$$

$\boxed{\text{markets are complete} \iff \mathbb{Q} \text{ is unique}}$

$\Rightarrow$ all claims are replicable.



need m - linearly independent assets to make a complete market.



$$V_0 = \alpha_0 A_0 + \beta_0 B_0$$

$$V_1 = \alpha_0 A_1 + \beta_0 B_1 = \alpha_1 A_1 + \beta_1 B_1$$

self-financing

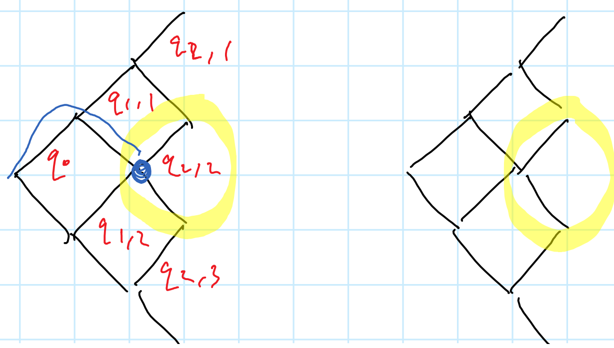
$$\alpha_{t-1} A_t + \beta_{t-1} B_t = \alpha_t A_t + \beta_t B_t , \quad \forall\, t > 0$$

An arbitrage strategy is $(\alpha_t, \beta_t)_{t \geq 0}$ s.t.

i) $V_0 = 0$

ii) $\exists\, t$ s.t. a) $\mathbb{P}(V_t \geq 0) = 1$

b) $\mathbb{P}(V_t > 0) > 0$



sigma-algebra generated by traded assets.

$q_{2,3}$

assets.

absence of arbitrage strategies

$\Longleftrightarrow$ $\exists$ $\mathbb{Q} \sim \mathbb{P}$ s.t. relative prices, w.r.t a numeraire, are martingales

$$\frac{A_t}{B_t} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_s}{B_s} \,\middle|\, \mathcal{F}_t \right],$$

$\forall\, s > t \geq 0$ & all traded assets A.

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CRR model

(Cox, Ross, Rubenstein)

$$A_n = A_{n-1}\, e^{\sigma\sqrt{\Delta t}\, x_n}, \quad A_0 > 0$$

$x_1, x_2, \ldots$ iid Bernoulli (± 1)

$$\mathbb{P}(x_k = 1) = \frac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right)$$

$$B_n = B_{n-1}\, e^{r\Delta t}, \quad B_0 = 1$$

$$\frac{A_{n-1}}{B_{n-1}} \begin{cases} \xrightarrow{q} \dfrac{A_{n-1}}{B_{n-1}}\, e^{\sigma\sqrt{\Delta t} - r\Delta t} \\[2ex] \phantom{\xrightarrow{q}} \dfrac{A_{n-1}}{B_{n-1}}\, e^{-\sigma\sqrt{\Delta t} - r\Delta t} \end{cases}$$

$$1 = e^{\sigma\sqrt{\Delta t} - r\Delta t}\, q + e^{-\sigma\sqrt{\Delta t} - r\Delta t}(1-q)$$

$$\Rightarrow \boxed{q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}} \in (0,1)$$

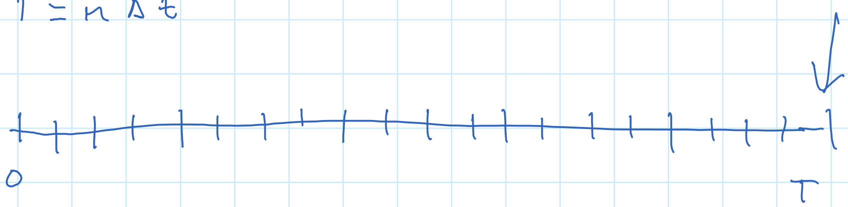
$$\sigma, r > 0, \quad r\Delta t < \sigma\sqrt{\Delta t}$$

$$r < \frac{\sigma}{\sqrt{\Delta t}}$$

NO ARBITRAAGE!

$$A_n = A_0\, e^{\underbrace{\sigma\sqrt{\Delta t}\sum_{m=1}^{n} x_m}_{X}}$$

$T = n\,\Delta t$



Characteristic Function $\cancel{\text{m.g.f.}}$ $\psi_n(u) = \mathbb{E}^{\mathbb{P}}\left[\, e^{iuX} \,\right]$

$$\psi_n(u) = \left(\mathbb{E}\left[\, e^{iu\sigma\sqrt{\Delta t}\, x_1} \,\right]\right)^n$$

$$\underbrace{\mathbb{E}^{\mathbb{P}}\left[\, e^{iu\sigma\sqrt{\Delta t}\, x_1} \right]}_{\ell(u)} \simeq \mathbb{E}^{\mathbb{P}}\left[\, 1 + iu\sigma\sqrt{\Delta t}\, x_1 - \tfrac{1}{2}u^2\sigma^2\Delta t\, x_1^2 + o(\Delta t) \right]$$

$$\mathbb{E}^{\mathbb{P}}[x_1] = p + (-1)(1-p) = 2p - 1$$

$$= \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}$$

$$\mathbb{E}^{\mathbb{P}}[x_1^2] = 1$$

$$\Rightarrow \ell(u) = 1 + iu\,\overbrace{(\mu - \tfrac{1}{2}\sigma^2)}^{\tilde{\mu}}\,\Delta t - \tfrac{1}{2}u^2\sigma^2\Delta t + o(\Delta t)$$

$$= 1 + \left(iu\tilde{\mu} - \tfrac{1}{2}\sigma^2u^2\right)\Delta t + o(\Delta t)$$

$$= e^{(iu\tilde{\mu} - \frac{1}{2}\sigma^2u^2)\Delta t} + o(\Delta t)$$

$$\psi_n(u) = e^{iuT\tilde{\mu} - \frac{1}{2}\sigma^2 T u^2} + o(\Delta t)$$

$$\longrightarrow e^{iu\,T\tilde{\mu} - \frac{1}{2}\sigma^2 T\, u^2}$$

$$\xrightarrow[n\to\infty]{} \quad e^{\,i u T \tilde{\mu} - \frac{1}{2}\sigma^2 T u^2}$$

↳ characteristic fn. of a normal r.v

$$X \underset{n\to\infty}{\overset{\mathbb{P}}{\sim}} \mathcal{N}\left(\tilde{\mu}\, T \; ; \; \sigma^2 T \right)$$

$$A_T = A_0\, e^{X} \overset{d}{=} A_0\, e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z}$$

$$Z \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$$

i.e. it is log-normal r.v.

$$q = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

$\rightarrow (\cancel{1} + r\Delta t) - (\cancel{1} - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2 \Delta t) + \dots$

$= \sigma\sqrt{\Delta t} + (r - \frac{1}{2}\sigma^2)\Delta t + \dots$

$\rightarrow (\cancel{1} + \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2\Delta t}) - (\cancel{1} - \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2\Delta t})$

$= 2\sigma\sqrt{\Delta t} + \dots$

$$= \frac{1}{2}\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t} \right) + \dots$$

$$p = \frac{1}{2}\left(1 + \frac{\mu - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t} \right)$$

$$X \underset{n\to\infty}{\overset{\mathbb{Q}}{\sim}} \mathcal{N}\left((r - \tfrac{1}{2}\sigma^2)\, T \; ; \; \sigma^2 T \right)$$

$$A_T = A_0\, e^{X} \overset{d}{=} A_0\, e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z}$$

$$Z \overset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$$

asset returns

$$\mathbb{E}^{\mathbb{P}}[A_T / A_0] = \mathbb{E}^{\mathbb{P}}\left[e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z} \right]$$

$$Z \overset{\mathbb{P}}{\sim} \mathcal{N}(0,1)$$

$$= e^{(\mu - \frac{1}{2}\sigma^2)T}\, \mathbb{E}^{\mathbb{P}}\left[e^{\sigma\sqrt{T}\, Z} \right]$$

$$\hookrightarrow \quad e^{\frac{1}{2}\sigma^2 T}$$

$$= \int_{-\infty}^{\infty} e^{uz} \cdot \frac{e^{-\frac{1}{2}z^2}}{\sqrt{2\pi}}\, dz$$

asset's return

$$\mathbb{E}^{\mathbb{P}}\left[\frac{A_T}{A_0}\right] = e^{\mu T}$$

$$\boxed{\mathbb{E}^{\mathbb{Q}}\left[\frac{A_T}{A_0}\right] = e^{rT}}$$

$$\frac{A_0}{B_0} = \mathbb{E}^{\mathbb{Q}}\left[\frac{A_T}{B_T}\right] \Rightarrow A_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT} A_T\right]$$

Asset as numeraire:

$$\frac{B_{n-1}}{A_{n-1}} \begin{cases} \dfrac{B_{n-1}\, e^{r\Delta t}}{A_{n-1}\, e^{\sigma\sqrt{\Delta t}}} \\[2ex] \dfrac{B_{n-1}\, e^{r\Delta t}}{A_{n-1}\, e^{-\sigma\sqrt{\Delta t}}} \end{cases}$$

$$1 = e^{r\Delta t - \sigma\sqrt{\Delta t}}\, q^A + e^{r\Delta t + \sigma\sqrt{\Delta t}}\, (1 - q^A)$$

$$\Rightarrow \quad \boxed{q^A = \frac{e^{-r\Delta t} - e^{\sigma\sqrt{\Delta t}}}{e^{-\sigma\sqrt{\Delta t}} - e^{\sigma\sqrt{\Delta t}}}} \quad \in (0,1)$$

$$= \frac{(\cancel{1} - r\Delta t) - (\cancel{1} + \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t) + \cdots}{(\cancel{1} - \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2\Delta t}) - (\cancel{1} + \sigma\sqrt{\Delta t} + \cancel{\tfrac{1}{2}\sigma^2\Delta t}) + \cdots}$$

$$= \frac{-\sigma\sqrt{\Delta t} - (r + \frac{1}{2}\sigma^2)\Delta t + \cdots}{-2\sigma\sqrt{\Delta t} + \cdots}$$

$$= \frac{1}{2}\left(1 + \frac{r + \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \cdots$$

$$q^B = \frac{1}{2}\left(1 + \frac{r - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right) + \cdots$$

$r + \sigma^2$

$$A_T \stackrel{d}{=} e^{(r + \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,Z}$$

$$Z \stackrel{Q^A}{\sim} \mathcal{N}(0,1)$$

$$\mathbb{E}^{Q^A}[A_T / A_0] = e^{(r+\sigma^2)T}$$

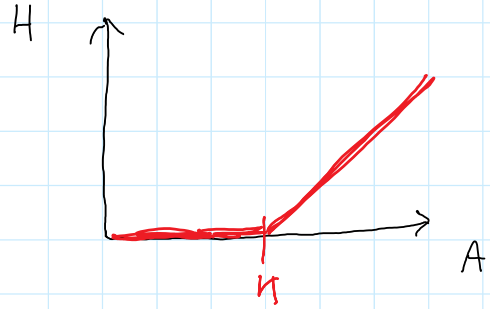
European option: pays $H(A_T)$ at T.

call option $H(x) = (x - K)_+$

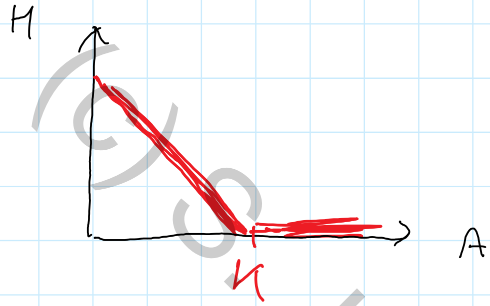
$:= \max(x - K;\ 0)$

H $\uparrow$

$$:= \max(x-K\,;\,0)$$



put option $H(x) = (K-x)_+$



$$\mathbb{1}_\omega = \begin{cases} 1 & \omega \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

$$H(x) = (x-K)_+ = (x-K)\,\mathbb{1}_{\{x>K\}}$$

$$= \underbrace{x\,\mathbb{1}_{x>K}}_{J} - \underbrace{K\,\mathbb{1}_{x>K}}_{L}$$

$$\frac{J_0}{B_0} = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{J_T}{B_T}\right]$$

$$\frac{L_0}{B_0} = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{L_T}{B_T}\right] = \mathbb{E}^{\mathbb{Q}^B}\left[\mathbb{1}_{A_T>K}\right] K e^{-rT}$$

$$= \mathbb{Q}^B(A_T>K)\; K\, e^{-rT}$$

$$\frac{J_0}{B_0} = \mathbb{E}^{\mathbb{Q}^B}\left[\frac{A_T\,\mathbb{1}_{A_T>K}}{e^{rT}}\right]$$

$$\frac{J_0}{} = \mathbb{E}^{\mathbb{Q}^A}\left[\frac{\cancel{A_T}\,\mathbb{1}_{A_T>K}}{}\right.$$

$$\frac{J_0}{A_0} = \mathbb{E}^{\mathbb{Q}^A}\left[\frac{\cancel{A_T}\, \mathbb{1}_{A_T > K}}{\cancel{A_T}} \right]$$

$$= \mathbb{Q}^A (A_T > K)$$

$$\boxed{V_0^{call} = A_0\, \mathbb{Q}^A (A_T > K) - K e^{-rT} \mathbb{Q}^B (A_T > K)}$$

$$\mathbb{Q}^B\left(A_0\, e^{(r - \frac{1}{2}\sigma^2) T + \sigma\sqrt{T}\, Z} > K \right)$$

$$Z \overset{\mathbb{Q}^B}{\sim} \mathcal{N}(0,1)$$

$$= \mathbb{Q}^B\left(Z > \frac{\log(K/A_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)$$

$$= \mathbb{Q}^B\left(Z < \underbrace{\frac{\log(A_0/K) + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{d_-} \right)$$

$$= \Phi (d_-)$$

$$\mathbb{Q}^A (A_T > K) = \Phi (d_+) \qquad d_+ = \frac{\log(A_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$\boxed{V_0^{call} = A_0\, \Phi(d_+) - K e^{-rT}\, \Phi(d_-)}$$

Black-Scholes Formula